

COMPACT QUOTIENT SPACES OF C^2 BY AFFINE TRANSFORMATION GROUPS

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The purpose of this paper is to classify the compact complex surfaces of the form C^2/G , where G is a properly discontinuous and fixed point free group of affine transformations of the two-dimensional complex vector space C^2 . Except for the use of some theorems on numerical characters of a compact complex surface, the method is mostly elementary.

§ 1 contains preliminary considerations on some properties of a fixed point free affine transformation group of C^2 . In § 2 we perform the classification. Denoting by b_1 the first Betti number of the quotient space $S = C^2/G$, we prove that if $b_1 = 4$ then S is a complex torus (Theorem 1), if $b_1 = 3$ then S is a fiber bundle of elliptic curves over an elliptic curve (Theorem 2), if $b_1 = 2$ then S is a hyperelliptic surface (Theorem 3), and if $b_1 = 1$ then S is an elliptic surface over the projective line with multiple singular fibers (Theorem 4).

1. A fundamental lemma

Let G denote a group of affine transformations of the two-dimensional complex vector space C^2 . Assume the action of G is (A) properly discontinuous, i.e., for any pair (K_1, K_2) of compact subsets in C^2 , the set $\{g \in G \mid gK_1 \cap K_2 \neq \emptyset\}$ is finite, and (B) fixed point free, i.e., for all $g \in G$, $g \neq 1$, g has no fixed points. Thus the quotient space C^2/G is a complex manifold of complex dimension 2. Finally we assume (C) C^2/G is compact. The problem is to classify the compact complex surfaces of the form C^2/G . In this section we prove a fundamental lemma for this purpose.

First of all, each element g of G is expressed by a 3×3 matrix:

$$g = \begin{pmatrix} a_{11}(g) & a_{12}(g) & b_1(g) \\ a_{21}(g) & a_{22}(g) & b_2(g) \\ 0 & 0 & 1 \end{pmatrix},$$

which acts on $C^2 = \{z \mid z = (z_1, z_2)\}$ by

Received June 28, 1972, and, in revised form, April 2, 1974. Research partially supported by an NSF grant. After this paper was submitted, the author was informed that the material in §1 and some of the preliminary material in §2 had been published by J. P. Fillmore and J. Scheuneman, *Fundamental groups of compact complete locally affine complex surfaces*, Pacific J. Math. **44** (1973) 487-496.

$$\begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} z'_1 \\ z'_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11}(g) & a_{12}(g) & b_1(g) \\ a_{21}(g) & a_{22}(g) & b_2(g) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}.$$

We put

$$* A(g) = \begin{pmatrix} a_{11}(g) & a_{12}(g) \\ a_{21}(g) & a_{22}(g) \end{pmatrix}, \quad b(g) = \begin{pmatrix} b_1(g) \\ b_2(g) \end{pmatrix}.$$

Note that $\det A(g) \neq 0$. Moreover, that g has no fixed points means the linear equation

$$(A(g) - I) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -b(g)$$

has no solution for $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where I denotes the 2×2 unit matrix. In particular,

$$(1) \quad \det(A(g) - I) = 0,$$

$$(2) \quad \text{if } b(g) = 0, \text{ then } g = 1.$$

For elements g and h of G we have

$$\begin{aligned} A(g^{-1}) &= A(g)^{-1}, & b(g^{-1}) &= -A(g)^{-1}b(g), \\ A(gh) &= A(g) \cdot A(h), & b(gh) &= A(g)b(h) + b(g). \end{aligned}$$

Next we consider the space $E(2, 1)$ of lines in \mathbb{C}^2 and the action of G on $E(2, 1)$. A line L is a subvariety of \mathbb{C}^2 defined by a linear equation $\alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 = 0$, $(\alpha_1, \alpha_2) \neq (0, 0)$. Let $E(2, 1)$ denote the set of lines in \mathbb{C}^2 . Two equations $\alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 = 0$ and $\alpha'_0 + \alpha'_1 z_1 + \alpha'_2 z_2 = 0$ represent the same line if and only if there exists a complex number $\lambda \neq 0$ such that $\alpha'_\nu = \lambda \alpha_\nu$, for $\nu = 0, 1, 2$. Hence we have a bijection

$$E(2, 1) \xrightarrow{\sim} \mathbb{P}^2 - \{p\}, \quad p = (1 : 0 : 0),$$

given by $L = \{(z_1, z_2) \mid \alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 = 0\} \mapsto (\zeta_0 : \zeta_1 : \zeta_2) = (\alpha_0 : \alpha_1 : \alpha_2)$, where \mathbb{P}^2 denotes the two-dimensional complex projective space with homogeneous coordinates $(\zeta_0 : \zeta_1 : \zeta_2)$. We identify $E(2, 1)$ with $\mathbb{P}^2 - \{p\}$ by this bijection. If we denote by $G(2, 1)$ the set of lines in \mathbb{C}^2 passing through the origin, then $G(2, 1)$ is the projective line \mathbb{P}^1 in $E(2, 1)$ defined by $\zeta_0 = 0$. We have a fibering $\pi : E(2, 1) \rightarrow G(2, 1)$ defined by $(\zeta_0 : \zeta_1 : \zeta_2) \mapsto (\zeta_1 : \zeta_2)$. Thus $E(2, 1)$ is a complex line bundle over $G(2, 1) = \mathbb{P}^1$ of degree 1. Since G is a group of affine transformations, G acts naturally on $E(2, 1)$. Take $L \in E(2, 1)$ which is represented by $\alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 = 0$. Then L is transformed by g to

the line $\alpha'_0 + \alpha'_1 z_1 + \alpha'_2 z_2 = 0$, where $\alpha'_0 = \alpha_0 + (\alpha_1, \alpha_2)b(g)$, $(\alpha'_1, \alpha'_2) = (\alpha_1, \alpha_2) \cdot A(g)^{-1}$. Since G acts as a group of bundle automorphisms, G acts on the base space $G(2, 1) = P^1 = \{(\zeta_1 : \zeta_2)\}$ by the formula

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \mapsto {}^t A(g)^{-1} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}.$$

For a point p of $G(2, 1)$, let $H_p = \{g \in G \mid gp = p\}$ be the isotropy subgroup of G at p .

Remark. Thus we get a representation of G into the group of one-dimensional projective linear transformations $PGL(1, C)$. The kernel is the subgroup $\{g \in G \mid A(g) = 1\}$, i.e., the group of translations.

Lemma 1.1. *There exists a point p_0 on $G(2, 1)$ for which $H_{p_0} = G$.*

Proof. Suppose for any point p , $H_p \subsetneq G$. Fix an element g which acts non-trivially on $G(2, 1)$. Note that the number of the fixed points of g on $G(2, 1) = P^1$ is 1 or 2.

Case I: g has only one fixed point p_1 . By a suitable coordinate transformation, we may assume that $p_1 = 0 = (1 : 0)$ and $g(\infty) = 1$, $\infty = (0 : 1)$, $1 = (1 : 1)$. In view of (1) we have $A(g^{-1}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. By assumption, there exists an element h such that $h(p_1) \neq p_1$. If we put ${}^t A(h)^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $c \neq 0$. On the other hand, $0 = \det(A(h^{-1}) - I) = (a - 1)(d - 1) - bc$ by (1). Thus we have $\det(A(h^{-1}g^{-1}) - I) = (a - 1)(d - 1) - bc - c = -c \neq 0$. This means gh has a fixed point on C^2 , a contradiction.

Case II: g has two fixed points p_1 and p_2 on $G(2, 1)$. By a suitable coordinate transformation, we may assume $p_1 = 0$ and $p_2 = \infty$. This implies that $A(g)^{-1} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \neq d$. On the other hand, $0 = \det(A(g)^{-1} - I) = (a - 1)(d - 1)$. By assumption there exist elements $g_i \notin H_{p_i}$, for $i = 1, 2$. Now we can divide our discussion into the following three cases.

(α) $g_1 \notin H_{p_2}$. Put ${}^t A(g_1)^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. Then $g_1(0) \neq 0$ and $g_1(\infty) \neq \infty$ imply that $b_1 c_1 \neq 0$. On the other hand, $0 = \det(A(g_1)^{-1} - I) = (a_1 - 1) \cdot (d_1 - 1) - b_1 c_1$. Put $\Delta = \det(A(g_1^{-1}g^{-1}) - I)$. Then we have

$$\Delta = (a - 1)(d_1 - 1) + (d - 1)(a_1 - 1),$$

where $(a - 1)(d - 1) = 0$ and $(a_1 - 1)(d_1 - 1) \neq 0$. Hence $\Delta \neq 0$, which means $g g_1$ has a fixed point on C^2 .

(β) $g_2 \notin H_{p_1}$. We can get a contradiction by the same argument as in case (α).

(γ) $g_1 \in H_{p_2}$ and $g_2 \in H_{p_1}$. We have $g_1 g_2 \notin H_{p_1}$ and $g_1 g_2 \notin H_{p_2}$, and this case is then reduced to case (α) if we replace g_1 by $g_1 g_2$. q.e.d.

By a suitable coordinate transformation, we may assume $p_0 = \infty, H_\infty = G$. Then for any element g of G , $A(g) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is a triangular matrix. Hence we get

Corollary. *The group G is solvable.*

From now on, we always assume $a_{21}(g) = 0$ for every $g \in G$.

Remarks. 1. In the proof of the lemma, we only used the fact that the action of G on C^2 is fixed point free. Moreover, from this fact we have either $a_{11}(g) = 1$ for all $g \in G$ or $a_{22}(g) = 1$ for all $g \in G$.

2. Every element g of G is compatible with the projection $(z_1, z_2) \mapsto z_2$ of C^2 onto the second factor U_2 . This suggests the fiber structure of C^2/G over U_2/G (see the proofs of Theorem 2 and 4).

2. Classification

We need some formulas for numerical characters of a compact complex surface. Denote by S a compact complex surface, i.e., a compact complex manifold of complex dimension 2, and by \mathcal{O} and Ω^ν , respectively, the sheaves over S of germs of holomorphic functions and holomorphic ν -forms. Define $h^{\nu, \mu} = \dim H^\mu(S, \Omega^\nu)$. The geometric genus p_g and the irregularity q of S are defined, respectively, by $p_g = h^{0,2}$ and $q = h^{0,1}$. By the duality theorem, $p_g = h^{0,2} = h^{2,0}$. Moreover, we denote by b_ν the ν -th Betti number, and by c_ν the ν -th Chern class of S . Among these numerical characters, the Noether formula due to Hirzebruch, Atiyah and Singer holds:

$$(3) \quad 12(p_g - q + 1) = c_1^2 + c_2.$$

Moreover a theorem of Kodaira [3, I, Theorem 3] says

$$(4) \quad \begin{array}{l} \text{if } b_1 \text{ is even, then } 2q = b_1 \text{ and } h^{1,0} = q; \\ \text{if } b_1 \text{ is odd, then } 2q = b_1 + 1 \text{ and } h^{1,0} = q - 1. \end{array}$$

Take an affine transformation group G of C^2 satisfying conditions (A), (B), and (C) in § 1. Note that G , being the fundamental group of a compact space, is finitely generated.

The following proposition is obvious.

Proposition 1. *If $H_p = G$ for every point p of $G(2, 1)$, i.e., if every element of G is a translation, then $S = C^2/G$ is a complex torus.*

From now on, we assume that there exists an element of G which is not a translation. We classify the cases as follows:

$$\exists g_0, a_{12}(g_0) \neq 0 \quad \begin{cases} \forall g, a_{11}(g) = a_{22}(g) = 1. & (\alpha) \\ \exists g_1, a_{11}(g_1) \neq 1. & (\gamma 1) \\ \exists g_2, a_{22}(g_2) \neq 1. & (\gamma 2) \end{cases}$$

$$\forall g, a_{12}(g) = 0, \quad \exists g_2, a_{22}(g_2) \neq 1. \tag{\beta}$$

Lemma 2.1. *Case $(\gamma 1)$ is reduced to case (β) .*

Proof. Take two elements g and h of G . Their commutator is given by

$$ghg^{-1}h^{-1} = \begin{pmatrix} 1, a_{12}(h)(a_{11}(g) - 1) - a_{12}(g)(a_{11}(h) - 1), * \\ 0, & 1 & , 0 \\ 0, & 0 & , 1 \end{pmatrix}.$$

Since $ghg^{-1}h^{-1}$ has no fixed points on C^2 , we have

$$a_{12}(h)(a_{11}(g) - 1) - a_{12}(g)(a_{11}(h) - 1) = 0.$$

By assumption, there exist g_0 and g_1 with $a_{12}(g_0) \neq 0$ and $a_{11}(g_1) \neq 1$. Thus there exists a nonzero complex number λ such that $a_{12}(g) - \lambda(a_{11}(g) - 1) = 0$ for any g . If we introduce new coordinates (z'_1, z'_2) of C^2 by $\begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, we see that case $(\gamma 1)$ is reduced to case (β) . q.e.d.

In view of this lemma, we may assume $a_{11}(g) = 1$ for any $g \in G$ in any case (cf. Remark 1 at the end of § 1).

Lemma 2.2. *Case $(\gamma 2)$ is reduced to case (β) if there exists a complex number λ such that, for any g ,*

$$(5) \quad a_{12}(g) + \lambda(a_{22}(g) - 1) = 0.$$

Proof. This can be done by applying the coordinate transformation $\begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

Thus in case $(\gamma 2)$, we assume that

(*) for any complex number λ , there exists an element g such that (5) does not hold.

Lemma 2.3. *In cases (β) and $(\gamma 2)$, the center C of G is given by*

$$C = \{g \in G \mid A(g) = I, b_2(g) = 0\}.$$

Proof. It is clear that an element g with $A(g) = I$ and $b_2(g) = 0$ is in C . Take an element g in C . For any element h of G , we have

$$(6) \quad \begin{aligned} a_{12}(g)(a_{22}(h) - 1) - a_{12}(h)(a_{22}(g) - 1) &= 0, \\ (a_{22}(g) - 1)b_2(h) - (a_{22}(h) - 1)b_2(g) &= 0, \\ a_{12}(g)b_2(h) - a_{12}(h)b_2(g) &= 0. \end{aligned}$$

We claim that $a_{22}(g) = 1$. In case $(\gamma 2)$, this is trivial in view of the assumption (*). In case (β) , this is proved as follows. Assume $a_{22}(g) \neq 1$ and put $\lambda =$

$b_2(g)/(a_{22}(g) - 1)$. Introducing new coordinates of C^2 by $(z'_1, z'_2) = (z_1, z_2 + \lambda)$, we see that we can assume $b_2(h) = 0$ for any $h \in G$. Then G acts on the line $z'_2 = 0$ effectively, and the action is properly discontinuous. Hence we have $G \subset Z \oplus Z$, where Z denotes the ring of integers. Thus C^2/G cannot be compact (see the following proposition).

Finally, the existence of an element g_2 with $a_{22}(g_2) \neq 1$ implies $a_{12}(g) = b_2(g) = 0$.

Proposition 2. *Let F be a free abelian group acting on C^2 freely and properly discontinuously. If the rank of F is less than or equal to 3, then the quotient space C^2/F cannot be compact.*

Proof. As C^2 is an acyclic space, we have an isomorphism

$$H^n(C^2/F, Z) \xrightarrow{\sim} H^n(F, Z), \quad n = 0, 1, \dots,$$

where $H^n(F, Z)$ denotes the n -th cohomology group of F with coefficients in the trivial F -module Z . Let r be the rank of F . Then the cohomology groups $H^n(F, Z)$ are isomorphic to the cohomology groups of the real r -torus T^r . If C^2/F were compact, we would have $H^4(C^2/F, Z) = Z$. On the other hand, $H^4(F, Z) = H^4(T^r, Z) = 0$ since $r \leq 3$, which is a contradiction.

Lemma 2.4. *For any $g \in G$, $a_{22}(g)$ is a root of unity.*

Proof. First we prove that $|a_{22}(g)| = 1$ for every $g \in G$. Assume there exists an element g with $|a_{22}(g)| \neq 1$. By taking its inverse, if necessary, we may assume $|a_{22}(g)| < 1$. The n -th power of g is given by

$$g^n = \begin{pmatrix} 1 & \alpha_n & \beta_n \\ 0 & \gamma_n & \delta_n \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} \alpha_n &= \frac{a_{22}(g)^n - 1}{a_{22}(g) - 1} \cdot a_{12}(g), \\ \beta_n &= nb_1(g) + \left(\frac{a_{22}(g)^n - 1}{(a_{22}(g) - 1)^2} - \frac{n}{a_{22}(g) - 1} \right) \cdot a_{12}(g)b_2(g), \\ \gamma_n &= a_{22}(g)^n, \quad \delta_n = \frac{a_{22}(g)^n - 1}{a_{22}(g) - 1} \cdot b_2(g). \end{aligned}$$

Put $\alpha = -a_{12}(g)/(a_{22}(g) - 1)$ and $\delta = -b_2(g)/(a_{22}(g) - 1)$. Then $\alpha_n \rightarrow \alpha$, $\gamma_n \rightarrow 0$, and $\delta_n \rightarrow \delta$ as $n \rightarrow +\infty$.

For any element h , we have

$$A(g^n h g^{-n}) = \begin{pmatrix} 1, \gamma_n^{-1}(\alpha_n(a_{22}(h) - 1) + a_{12}(h)) \\ 0, a_{22}(h) \end{pmatrix},$$

$$b(g^n h g^{-n}) = \begin{pmatrix} -\gamma_n^{-1} \delta_n (\alpha_n (a_{22}(h) - 1) + a_{12}(h)) + \alpha_n b_2(h) + b_1(h) \\ \gamma_n b_2(h) - \delta_n (a_{22}(h) - 1) \end{pmatrix}.$$

Thus

$$g^n h g^{-n} \begin{pmatrix} z_1 \\ \delta \end{pmatrix} \rightarrow \begin{pmatrix} z_1 + \varepsilon(h) \\ \delta \end{pmatrix}, \quad \text{as } n \rightarrow +\infty,$$

where $\varepsilon(h) = \delta(\alpha(a_{22}(h) - 1) + a_{12}(h)) + \alpha b_2(h) + b_1(h)$.

Choose positive numbers c_1 and c_2 so that $|\varepsilon(h)| < c_1$. Consider the compact set K in C^2 defined by

$$K = \{(z_1, z_2) \mid |z_1| \leq c_1 \text{ and } |z_2 - \delta| \leq c_2\}.$$

Since $g^n h g^{-n}(0, \delta)$ converges to the point $(\varepsilon(h), \delta)$ as $n \rightarrow +\infty$, $g^n h g^{-n}(0, \delta) \in K$ for any large n . Since the action of G on C^2 is properly discontinuous, some positive power of g should commute with h . Moreover, since G is finitely generated, some power g^N of g should be contained in the center C . Hence we have $a_{22}(g)^N = 1$ by Lemma 2.3, which is a contradiction. Thus we have proved $|a_{22}(g)| = 1$ for any $g \in G$.

Since each entry of the matrix $g^n h g^{-n}$ remains bounded as n tends to infinity, by a similar argument as above we can prove $a_{22}(g)^n = 1$ for a positive integer n . q.e.d.

Let G^* be the normal subgroup of G defined by $G^* = \{g \in G \mid a_{22}(g) = 1\}$. Since G is finitely generated, Lemma 2.4 implies G/G^* is finite. Moreover, G^* is a nilpotent group. Thus we have

Corollary. *The group G^* is a nilpotent subgroup of G of finite index.*

Lemma 2.5. *The first Betti number b_1 of the quotient space $S = C^2/G$ is given by*

$$b_1 = \begin{cases} 4 \text{ or } 3, & \text{in case } (\alpha), \\ 2, & \text{in case } (\beta), \\ 2 \text{ or } 1, & \text{in case } (\gamma 2). \end{cases}$$

Proof. First we note that $\partial/\partial z_1$ is a nonvanishing G -invariant holomorphic vector field on C^2 . Hence by a theorem of Bott [1], we have $c_1^2 = c_2 = 0$ in each case. Next we find the number of linearly independent G -invariant holomorphic forms on C^2 . The pullbacks $g^* dz_i$ of dz_i , $i = 1, 2$, by an element g of G are given by $g^* dz_1 = dz_1 + a_{12}(g) dz_2$ and $g^* dz_2 = a_{22}(g) dz_2$. Thus we have $g^*(dz_1 \wedge dz_2) = a_{22}(g) dz_1 \wedge dz_2$.

Case (α) . Since $a_{22}(g) = 1$ for every g in G , a holomorphic 2-form $f(z) dz_1 \wedge dz_2$ on C^2 is G -invariant if and only if f is G -invariant. If f is G -invariant, f is considered to be a holomorphic function on the quotient space C^2/G , which is compact. Thus f is a constant, so that the geometric genus p_g

of $S = \mathbb{C}^2/G$, which is equal to the number of linearly independent holomorphic 2-forms on S , is equal to 1. Since the Noether formula (3) implies $q = 2$, by (4) we have $b_1 = 4$ or 3.

Case (β). Since $a_{12}(g) = 0$ for every g in G , the subgroup $G^* = \{g \in G \mid a_{22}(g) = 1\}$ of G consists of translations. Moreover, by the corollary to Lemma 2.4, the quotient space $T = \mathbb{C}^2/G^*$ is a finite unramified covering of S , which is compact. Thus T is a complex torus. Any G^* -invariant holomorphic 2-form on \mathbb{C}^2 is of the form $cdz_1 \wedge dz_2$ with c a constant. Since we have an element g_2 in G with $a_{22}(g_2) \neq 1$, no holomorphic 2-form on \mathbb{C}^2 is G -invariant, so that $p_g = 0$. Moreover, any G^* -invariant holomorphic 1-form on \mathbb{C}^2 is of the form $adz_1 + b dz_2$ with a and b constants. Since $g_2^*(adz_1 + b dz_2) = adz_1 + ba_{22}(g_2)dz_2$, the scalar multiples of dz_1 are the only G -invariant holomorphic 1-forms on \mathbb{C}^2 , which means $h^{1,0} = 1$. Therefore (3) and (4) imply $b_1 = 2$.

Case (γ 2). Consider $G^* = \{g \in G \mid a_{22}(g) = 1\}$. The quotient space $S^* = \mathbb{C}^2/G^*$ is a finite unramified covering of $S = \mathbb{C}^2/G$ and is a surface of case (α). As is seen in case (α), any G^* -invariant holomorphic 2-form on \mathbb{C}^2 is of the form $cdz_1 \wedge dz_2$ with c a constant. Since there is an element g_2 in G with $a_{22}(g_2) \neq 1$, we have $p_g = 0$, and therefore $q = 1$ by (3). Hence (4) implies $b_1 = 2$ or 1.

Theorem 1. *If $b_1 = 4$, then $S = \mathbb{C}^2/G$ is a complex torus.*

Proof. If G consists of only translations, the theorem is obvious. Thus we consider case (α) with $b_1 = 4$. From the assumption, we have $h^{1,0} = 2$. Let φ and ψ denote linearly independent G -invariant holomorphic 1-forms on \mathbb{C}^2 , and write $\varphi = \varphi_1(z)dz_1 + \varphi_2(z)dz_2$ and $\psi = \psi_1(z)dz_1 + \psi_2(z)dz_2$. Conditions for φ and ψ to be G -invariant are given by

$$(8) \quad \varphi_1(gz) = \varphi_1(z), \quad \psi_1(gz) = \psi_1(z),$$

$$(9) \quad \varphi_2(gz) = \varphi_2(z) - \varphi_1(gz)a_{12}(g), \quad \psi_2(gz) = \psi_2(z) - \psi_1(gz)a_{12}(g),$$

for any $g \in G$. From (8), we have $\varphi_1(z) = \varphi_1$ and $\psi_1(z) = \psi_1$ are constants, so that (9) reduces to

$$(10) \quad \varphi_2(gz) = \varphi_2(z) - \varphi_1 a_{12}(g), \quad \psi_2(gz) = \psi_2(z) - \psi_1 a_{12}(g).$$

Since $\varphi \wedge \psi = (\varphi_1\psi_2(z) - \psi_1\varphi_2(z))dz_1 \wedge dz_2$ is a G -invariant holomorphic 2-form on \mathbb{C}^2 , $\varphi_1\psi_2(z) - \psi_1\varphi_2(z) = c$ is a constant. We have $\psi_1\varphi - \varphi_1\psi = (\psi_1\varphi_2(z) - \varphi_1\psi_2(z))dz_2 = cdz_2$. If $c = 0$, we would have $\varphi_1 = \psi_1 = 0$, and then $\varphi_2(z)$ and $\psi_2(z)$ would be constant by (10), which is a contradiction. Hence $c \neq 0$. Consider the Albanese variety A of $S = \mathbb{C}^2/G$. Since A is a complex torus whose lattice Γ is generated by the periods of φ and ψ on four free generators for $H_1(S, \mathbb{Z})$, we have a canonical mapping $\Phi: S \rightarrow A$ defined by $\Phi(z) = \left(\int^z \varphi, \int^z \psi \right) \pmod{\Gamma}$ for $z \in S$. The Jacobian of Φ is given by $\varphi_1\psi_2(z) -$

$\psi_1\varphi_2(z) = c$, so that Φ is an unramified covering mapping. Hence $S = C^2/G$ is a complex torus.

Example. Consider the group G generated by four elements :

$$g_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, by a suitable coordinate transformation φ , say $\varphi(z_1, z_2) = (z_1 - \frac{1}{2}z_2^2, z_2)$, G is transformed into a group of translations. Moreover, $\varphi g_i \varphi^{-1}, i = 1, \dots, 4$, are linearly independent over \mathbf{R} . Thus C^2/G is a complex torus.

Theorem 2. *If $b_1 = 3$, then $S = C^2/G$ is a fiber bundle of elliptic curves over an elliptic curve.*

Proof. Take two elements g and h of G . Their commutator is given by

$$ghg^{-1}h^{-1} = \begin{pmatrix} 1 & 0 & a_{12}(g)b_2(h) - a_{12}(h)b_2(g) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $G^{(1)} = [G, G]$ be the commutator group of G . Then we have the following exact sequence :

$$(11) \quad 1 \longrightarrow G^{(1)} \longrightarrow G \xrightarrow{\varphi} H_1(S, \mathbf{Z}) \longrightarrow 0,$$

where $S = C^2/G$. Note that for any element g of $G^{(1)}$, $A(g) = I$ and $b_2(g) = 0$ and that $G^{(1)}$ is commutative.

Let U_1 and U_2 denote the first and second factors of the product C^2 . Then $G^{(1)}$ acts on U_1 effectively as a group of translations. Moreover, since the action of $G^{(1)}$ on C^2 is "parallel" to the z_1 -axis, we see that $G^{(1)}$ acts on U_1 properly discontinuously. Hence $G^{(1)}$ is a subgroup of $\mathbf{Z} \oplus \mathbf{Z}$.

(i) First we assume $G^{(1)} = 0$. Then we have $G = H_1(S, \mathbf{Z})$. The free part F of G is a free abelian group of rank 3. The quotient space C^2/F , being a finite covering of C^2/G , is compact, which is a contradiction (see Proposition 2).

(ii) Secondly we assume $G^{(1)} = \mathbf{Z}$. Let h_0 be a generator of the infinite cyclic group $G^{(1)}$, γ_1, γ_2 , and γ_3 generators of the free part of $H_1(S, \mathbf{Z})$, and τ_1, \dots, τ_t generators of the torsion part of $H_1(S, \mathbf{Z})$. Choose elements $h_i, i = 1, 2, 3$, and $k_j = 1, \dots, t$, of G so that $\varphi(h_i) = \gamma_i$ and $\varphi(k_j) = \tau_j$. Then G is generated by $h_0, h_1, h_2, h_3, k_1, \dots, k_t$.

Lemma 2.6. *Let g be an element of G . If $\varphi(g)$ is a torsion element, then $b_2(g) = 0$.*

Proof. The condition implies that some positive power g^n of g is contained in $G^{(1)}$. Hence we have $0 = b_2(g^n) = nb_2(g)$.

Lemma 2.7. *For any element g of G there exist integers $n_i, i = 1, 2, 3$, such that*

$$b_2(g) = \sum_{i=1}^3 n_i b_2(h_i) .$$

Proof. Since $b_2(gh) = b_2(g) + b_2(h)$ for any two elements g and h of G , the lemma follows from Lemma 2.6. q.e.d.

Consider the natural action of G on the second factor U_2 of C^2 , which is given by $g: z_2 \mapsto z_2 + b_2(g)$ for $g \in G$, and let G_1 denote the kernel of the action. Since G is free on C^2 , if $b_2(g) = 0$ then $a_{12}(g) = 0$. Thus an element g of G is contained in G_1 if and only if $b_2(g) = a_{12}(g) = 0$.

Lemma 2.8. *G/G_1 acts properly discontinuously on U_2 .*

Proof. Since the commutator group $G^{(1)}$ is generated by h_0 , there exists an integer n_{ij} for each pair $(h_i, h_j), i, j = 1, 2, 3$, such that

$$(12) \quad a_{12}(h_i)b_2(h_j) - a_{12}(h_j)b_2(h_i) = n_{ij}b_1(g) .$$

From (12), we get

$$(13) \quad n_{12}b_2(h_3) + n_{23}b_2(h_1) + n_{31}b_2(h_2) = 0 .$$

Assume $n_{12} = n_{23} = n_{31} = 0$. Then G should be commutative, which is a contradiction. Therefore at least one of n_{12}, n_{23} or n_{31} is nonzero, and we get a nontrivial linear relation (13) among $b_2(h_i)$ with integer coefficients. This fact, together with Lemma 2.7, implies the lemma. q.e.d.

Now we have $C^2/G = (C^2/G_1)/(G/G_1)$, where $C^2/G_1 = (U_1/G_1) \times U_2$. Since G/G_1 acts properly discontinuously on U_2 , C^2/G is a fiber bundle over the one-dimensional complex manifold $U_2/(G/G_1)$ with fiber U_1/G_1 . Hence U_1/G_1 and $U_2/(G/G_1)$ are compact. Moreover, since G_1 and G/G_1 act on U_1 and U_2 respectively as groups of translations, U_1/G_1 and $U_2/(G/G_1)$ are elliptic curves.

(iii) Finally, we assume $G^{(1)} = Z \oplus Z$. We have $C^2/G = (C^2/G^{(1)})/(G/G^{(1)})$. Since $G^{(1)} = Z \oplus Z$ acts trivially on U_2 , $C^2/G^{(1)} = (U_1/G^{(1)}) \times U_2$ is the product of the elliptic curve $U_1/G^{(1)}$ and U_2 . Let Γ denote the kernel of the natural action of $G/G^{(1)}$ on U_2 . Since $G/G^{(1)}$ acts properly discontinuously on $(U_1/G^{(1)}) \times U_2$, whose first factor is compact, $(G/G^{(1)})/\Gamma$ acts properly discontinuously on U_2 . Now as in case (ii), take elements h_1, h_2 , and h_3 of G such that $\varphi(h_1), \varphi(h_2)$, and $\varphi(h_3)$ generate the free part of $H_1(S, Z)$. Then $G^{(1)}$

is generated by $h_i h_j h_i^{-1} h_j^{-1} = \begin{pmatrix} 1 & 0 & \omega_{ij} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, i, j = 1, 2, 3$, where $\omega_{ij} = a_{12}(h_i)$

$\cdot b_2(h_j) - a_{12}(h_j)b_2(h_i)$. On the other hand, since $(G/G^{(1)})/\Gamma$ acts on U_2 properly discontinuously, we have a nontrivial relation:

$$(14) \quad \sum_{i=1}^3 n_i b_2(h_i) = 0 ,$$

where $n_i, i = 1, 2, 3$, are integers with $(n_1, n_2, n_3) \neq (0, 0, 0)$. Note that (14) implies $\sum_{i=1}^3 n_i a_{12}(h_i) = 0$. Thus we have the following equalities:

$$(15) \quad n_1 \omega_{12} - n_3 \omega_{23} = 0 , \quad n_2 \omega_{23} - n_1 \omega_{31} = 0 , \quad n_3 \omega_{31} - n_2 \omega_{12} = 0 .$$

Since $(n_1, n_2, n_3) \neq (0, 0, 0)$, (15) implies that $\text{rank } G^{(1)} \leq 1$, which is a contradiction. This completes the proof of Theorem 2.

A compact complex surface S is said to be an *elliptic surface* if there exists a holomorphic mapping Ψ of S onto a nonsingular curve Δ such that the inverse image $\Psi^{-1}(u)$ of any general point $u \in \Delta$ is an elliptic curve. For the theory of elliptic surfaces we refer to Kodaira [2]. Let $\Psi : S \rightarrow \Delta$ be a (holomorphic) fiber bundle of elliptic curves over an elliptic curve Δ , and assume that the first Betti number b_1 of S is equal to 3. Then the functional invariant of S is constant and the homological invariant of S is trivial [2, II, § 7], [4, p. 470]. Thus the basic member B is trivial; $B = C \times \Delta$, where C denotes the typical fiber of $S \rightarrow \Delta$. Hence the canonical bundle K of S is simply given by $K = \Psi^*(\kappa)$, where κ denotes the canonical bundle of Δ , [3, I, Theorem 12]. Since κ is trivial, so is K . Therefore, by Theorem 19 in [3, I], S is biholomorphic to a quotient space of C^2 by an affine transformation group G , which is generated by four elements g_1, g_2, g_3 and g_4 with a fundamental relation $g_3 g_4 = g_2^m g_1 g_3$, where m is a positive integer.

The fiber bundles over an elliptic curve Δ with fiber an elliptic curve C whose homological invariants are trivial are described as follows. First we express C as a quotient group: $C = C/\Gamma$, where Γ denotes a discrete subgroup of C generated by 1 and ω , $\text{Im } \omega > 0$, and for any $\zeta \in C$ we denote by $[\zeta]$ the corresponding element of $C = C/\Gamma$. We have the following sheaf exact sequence over Δ

$$0 \rightarrow \Gamma \rightarrow \mathcal{O} \rightarrow \mathcal{O}(C) \rightarrow 0 ,$$

where \mathcal{O} and $\mathcal{O}(C)$ denote the sheaves of germs of holomorphic functions and holomorphic mappings into C respectively. We have the corresponding cohomology exact sequence

$$\dots \rightarrow H^1(\Delta, \mathcal{O}) \xrightarrow{h} H^1(\Delta, \mathcal{O}(C)) \xrightarrow{c} H^2(\Delta, \Gamma) \rightarrow 0 .$$

Any fiber bundle S over Δ with fiber C whose homological invariant is trivial is written in the form $(C \times \Delta)^\gamma$, for some $\gamma \in H^1(\Delta, \mathcal{O}(C))$, [2, II, Theorem 10.1],

[4, p. 470]. Moreover, $S = (C \times \Delta)^{\eta}$ is a deformation of $S' = (C \times \Delta)^{\eta'}$ if the characteristic classes are the same; $c(\eta) = c(\eta')$, [2, III, Theorem 11.4]. The first Betti number b_1 of $S = (C \times \Delta)^{\eta}$ is 4 or 3 according as $c(\eta) = 0$ or $c(\eta) \neq 0$, [2, III, Theorem 11.9]. For each element $\gamma \in H^2(\Delta, \Gamma) = \Gamma \xrightarrow{\sim} \mathbb{Z} \oplus \mathbb{Z}$, we can construct a bundle S_{γ} with characteristic class γ as follows (cf. [3, II, p. 684]). Take a point p on Δ , and let z be a local coordinate with center p and $U = \{z \mid |z| < \varepsilon\}$ a small disk around p . S_{γ} is defined by $S_{\gamma} = U \times C \cup (\Delta - p) \times C$, where $(z, [\zeta]) \in U \times C$ and $(z, [\zeta']) \in (\Delta - p) \times C$ are identified if and only if $[\zeta'] = [\zeta + (\gamma/2\pi i) \log z]$. Thus any fiber bundle S over Δ with fiber C with $b_1 = 3$ is a deformation of S_{γ} for some $\gamma \in \Gamma$, $\gamma \neq 0$. If $\gamma = h + k\omega$, h and $k \in \mathbb{Z}$, we have $H_1(S, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_m$, where $m = (h, k)$.

A hyperelliptic surface is a fiber bundle of elliptic curves over an elliptic curve with $b_1 = 2$. For the classification of hyperelliptic surfaces we refer to [4].

Theorem 3. *If $b_1 = 2$, then $S = C^2/G$ is a hyperelliptic surface.*

Proof. By the characterization (D) in [4, p. 476] of hyperelliptic surfaces.

Remark. S is algebraic as $p_g = 0$ and b_1 is even [3, I. Theorem 10]. We can also prove (A), (B) or (C) in [4, p. 476] directly.

Theorem 4. *If $b_1 = 1$, then $S = C^2/G$ has the following structure:*

- (**)
- (1) S is an elliptic surface over the projective line \mathbb{P}^1 ,
 - (2) S has no singular fibers over the base curve \mathbb{P}^1 other than multiple fibers of the form $m\Theta$, where Θ is a nonsingular elliptic curve and m the multiplicity (type ${}_mI_0$ in [2]),
 - (3) the multiplicities m_i of the multiple fibers $m_i\Theta_i$, $i = 1, \dots, r$, of S satisfy the equality $\sum_{i=1}^r (1 - 1/m_i) = 2$.

Proof. Consider the normal subgroup $G^* = \{g \in G \mid a_{22}(g) = 1\}$ of G . By the corollary to Lemma 2, 4, G/G^* is finite. We have $C^2/G = (C^2/G^*)/(G/G^*)$. The surface $S^* = C^2/G^*$ is compact and is a surface of case (α). Thus the first Betti number b_1^* of S^* is either 3 or 4. If b_1^* were equal to 4, then by Theorem 1, S^* would be a complex torus, which is a Kähler manifold. Thus the finite quotient space $S = S^*/(G/G^*)$ is also a Kähler manifold, which is a contradiction since the first Betti number of S is odd. Hence $b_1^* = 3$. By Theorem 2, S^* is a fiber bundle of elliptic curves over an elliptic curve Δ^* . Let G_1^* be the kernel of the natural action of G^* on the second factor U_2 of C^2 . Then as is seen in the proof of Theorem 2, the base curve Δ^* is the quotient space $U_2/(G^*/G_1^*)$, and the typical fiber of the fiber bundle $S^* \rightarrow \Delta^*$ is the quotient space U_1/G_1 , where U_1 denotes the first factor of C^2 . For $z = (z_1, z_2) \in C^2$ and $g \in G$, the second component of gz is given by $a_{22}(g)z_2 + b_2(g)$ and depends only on z_2 . Hence G acts naturally on U_2 , which means that the action of G/G^* on the fiber bundle $S^* \rightarrow \Delta^*$ is fiber preserving. We have the following commutative diagram:

$$\begin{array}{ccc}
 S^* & \xrightarrow{\Pi} & S = S^*/(G/G^*) \\
 \Psi^* \downarrow & & \downarrow \Psi \\
 \Delta^* & \xrightarrow{\pi} & \Delta = \Delta^*/(G/G^*) .
 \end{array}$$

Since each element (different from the identity) of the group G/G^* is represented by an element g of G with $a_{22}(g) \neq 1$, the action of G/G^* on Δ^* is effective. Moreover, the action is properly discontinuous since the projection map Ψ^* is proper. Thus G/G^* is a finite cyclic group acting on the elliptic curve Δ^* with fixed points, and the quotient space $\Delta^*/(G/G^*)$ is biholomorphic to the projective line. For $z_1 \in U_1, z_2 \in U_2$, and $g \in G$, we denote by $[z_1], [z_2]$ and $[g]$ the corresponding points in $U_1/G_1^*, U_2/(G/G_1^*)$ and G/G^* , respectively. If a point p on Δ^* is not a fixed point of G/G^* , the fiber $\Psi^{-1}(\pi(p))$ is biholomorphic to the elliptic curve U_1/G_1^* . Consider a fixed point $p = [z_2^*]$ of G/G^* on Δ^* , and let $[g^o]$ be a generator of the isotropy subgroup $(G/G^*)_p$ of G/G^* at p and m the order of $[g^o]$. The group $(G/G^*)_p$ acts on the fiber $\Psi^{*-1}(p) = U_1/G_1^*$ by $[z_1] \mapsto [z_1 + a_{12}(g^o)z_2^* + b_1(g^o)]$. This action is effective since otherwise some power of $[g^o]$ would have fixed points on S^* . Thus we get a multiple fiber $m\theta, \theta \xrightarrow{\sim} (U_1/G_1^*)/(G/G^*)_p$, of type mI_0 in the elliptic surface $\Psi: S \rightarrow \Delta$ over the point $\pi(p)$. Moreover, the mapping π is a ramified covering map with ramification exponent m at p . Hence the Hurwitz formula impiles the equality in (3).

Remarks. 1. For the structure of a neighborhood of a multiple fiber of type mI_0 , see [3, II, p. 685].

2. As is seen in the proof of Theorem 4, G/G^* is a finite cyclic group acting effectively on an elliptic curve with fixed points. Thus the order of G/G^* is 2, 3, 4 or 6.

3. Let S be a complex surface with the property (**). Then the first Betti number b_1 of S is either 2 or 1, [3, II, p. 686]. Moreover, S admits a fiber bundle S^* of elliptic curves over an elliptic curve as an unramified covering [3, II, p. 690], [4, p. 476]. If $b_1 = 2$, then S is a hyperelliptic surface [4, p. 476(C)], and S^* is a complex torus. If $b_1 = 1$, then the first Betti number b_1^* of S^* is 3, and S^* is a quotient space of C^2 by an affine transformation group (see p. 239). The canonical bundle of S^* is trivial. Thus in both cases, S is a quotient space of C^2 by an affine transformation group [3, II, § 11, especially Theorem 39].

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